4210. Proposed by Van Khea and Leonard Giugiuc.
Let ABC be a triangle in which the circumcenter lies on the incircle. Furthermore,
let $BC = a, CA = b$ and $AB = c$.
For which triangles does the expression $\dfrac{a+b+c}{\sqrt[3]{abc}}$ attain its minimum?

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Let I and O be incenter and circumcenter, respectively. Since by Euler's Theorem $OI^2 = R^2 - 2Rr$
and O lies on the incircle then $OI = r$ and, therefore, $R^2 - 2Rr = r^2 \iff R = \left(\sqrt{2}+1\right) r$.
Since for any non-acute triangle holds inequality $R \geq \left(\sqrt{2}+1\right) r$ (Emmerich's Inequality [1]), where
equality occurs iff ABC is a right isosceles triangle then triangle ABC can't be obtuse (because then
$R > \left(\sqrt{2}+1\right) r)$ and, therefore, further we can assume that triangle ABC is non-obtuse, that is
$\cos A \cos B \cos C \geq 0 \iff s \geq 2R + r$ (because $\cos A \cos B \cos C = \dfrac{s^2 - (2R+r)^2}{4R^2}$).
Since in the triangle $ABC,$ by condition, holds $R = \left(\sqrt{2}+1\right) r$ then for this triangle we have
$$\cos A \cos B \cos C \geq 0 \iff s \geq 2\left(\sqrt{2}+1\right) r + r = \left(2\sqrt{2}+3\right) r \iff \frac{s}{r} \geq 2\sqrt{2}+3,$$
and, therefore, $\dfrac{(a+b+c)^3}{abc} = \dfrac{8s^3}{4Rrs} = \dfrac{2s^2}{Rr} = \dfrac{2s^2}{\left(\sqrt{2}+1\right) r^2} \geq \dfrac{2\left(2\sqrt{2}+3\right)^2}{\left(\sqrt{2}+1\right)} = 10\sqrt{2}+14.$
Noting that in inequality $\dfrac{s}{r} \geq 2\sqrt{2}+3$ equality occurs iff $\cos A \cos B \cos C = 0$ we can conclude that
lower bound $10\sqrt{2}+14$ can be attained only in right angled triangle.
We will prove that there is right triangle ABC such that $\dfrac{s}{r} = 2\sqrt{2}+3, R = \left(\sqrt{2}+1\right) r$ and $C = 90^\circ$.
Since $2R = a + b - 2r$ and $ab = 2F$ we obtain $a + b = 2(R + r) = 2\left(\left(\sqrt{2}+1\right) r + r\right) = 2\sqrt{2}\left(\sqrt{2}+1\right) r$
and $ab = 2sr = 2\left(2\sqrt{2}+3\right) r^2 = 2\left(\sqrt{2}+1\right)^2 r^2$.Hence $(a-b)^2 = (a+b)^2 - 4ab = 0.$
Thus, $a = b = \sqrt{2}\left(\sqrt{2}+1\right) r, c = 2\left(\sqrt{2}+1\right) r$.
Therefore, $\min \dfrac{a+b+c}{\sqrt[3]{abc}} = \sqrt{10\sqrt{2}+14} = \sqrt[3]{2} + \sqrt[6]{32}$ and can be attained in any isosceles
right triangle.

1. Recent Advances in Geometric Inequalities, D. S. Mitrinovic, J. Pecaric, V. Volenec, p.251.